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Totally geodesic submanifolds in Riemannian symmetric spaces

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In the first part of this expository article, the most important constructions and classification results concerning totally geodesic submanifolds in Riemannian symmetric spaces are summarized. In the second part, I describe the results of my classification of the totally geodesic submanifolds in the Riemannian symmetric spaces of rank 2.

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1. Totally geodesic submanifolds

A submanifold M' of a Riemannian manifold M is called *totally geodesic*, if every geodesic of M' is also a geodesic of M. In this article, we will discuss totally geodesic submanifolds in Riemannian symmetric spaces; in such spaces, a connected, complete submanifold is totally geodesic if and only if it is a symmetric subspace.

There are several important construction principles for totally geodesic submanifolds in Riemannian symmetric spaces M. First, we note that the connected components of the fixed point set of any isometry f of M are totally geodesic submanifolds (this is in fact true in any Riemannian manifold, see Ref. 11, Theorem II.5.1, p. 59). This construction principle is especially important in the case where f is involutive (i.e. $f \circ f = \mathrm{id}_M$); the totally geodesic submanifolds resulting in this case are called *reflective* submanifolds; they have been studied extensively, for example by Leung (see below).

Further constructions of totally geodesic submanifolds in Riemannian symmetric spaces of compact type M were introduced by Chen and Nagano:[2,4] For $p \in M$, the connected components $\neq \{p\}$ of the fixed point set of the geodesic reflection of M at p are called *polars* or M_+-

submanifolds of M; note that they are in particular reflective submanifolds of M. A *pole* of M is a polar which is a singleton. It has been shown by Chen/Nagano[4] that for every polar M_+ of M and every $q \in M_+$ there exists another reflective submanifold M_- of M with $q \in M_-$ and $T_q M_- = (T_q M_+)^\perp$; M_- is called a *meridian* or M_-*-submanifold* of M. For $p_1, p_2 \in M$, a point $q \in M$ is called a *center point* between p_1 and p_2 if there exists a geodesic joining p_1 with p_2 so that q is the middle point on that geodesic. If p_2 is a pole of p_1, then the set $C(p_1, p_2)$ of center points between p_1 and p_2 is called the *centrosome* of p_1 and p_2; its connected components are totally geodesic submanifolds of M (see Ref. 2, Proposition 5.1).

Moreover, every symmetric space of compact type can be embedded in its transvection group as a totally geodesic submanifold: Let $M = G/K$ be such a space, then there exists an involutive automorphism σ of G so that $\mathrm{Fix}(\sigma)^0 \subset K \subset \mathrm{Fix}(\sigma)$. Because of this property, the *Cartan map*

$$f : G/K \to G,\ gK \mapsto \sigma(g) \cdot g^{-1}$$

is a well-defined covering map onto its image, which turns out to be a totally geodesic submanifold of G. If M is a "bottom space", i.e. there exists no non-trivial symmetric covering map with total space M, then we have $K = \mathrm{Fix}(\sigma)$, and therefore f is an embedding. In this setting f is called the *Cartan embedding* of M.

It is a significant and interesting problem to determine all totally geodesic submanifolds in a given symmetric space. Because totally geodesic submanifolds are rigid (i.e. if M_1', M_2' are connected, complete totally geodesic submanifolds of M with $p \in M_1' \cap M_2'$ and $T_p M_1' = T_p M_2'$, then we already have $M_1 = M_2$), they can be classified by determining those linear subspaces $U \subset T_p M$ which occur as tangent spaces of totally geodesic submanifolds of M.

The elementary answer to the latter problem is the following: There exists a totally geodesic submanifold of M with a given tangent space $U \subset T_p M$ if and only if U is *curvature invariant* (i.e. we have $R(u, v)w \in U$ for all $u, v, w \in U$, denoting by R the Riemannian curvature tensor of M).

Therefore the classification of totally geodesic submanifolds of M reduces to the purely algebraic problem of the classification of curvature invariant subspaces of $T_p M$. However, because of the algebraic complexity of the curvature tensor, classifying the curvature invariant subspaces is by no means an easy task, and therefore also the classification of totally geodesic submanifolds remains a signification problem.

This problem has been solved for the Riemannian symmetric spaces of rank 1 by Wolf in Ref. 19, §3. Chen/Nagano claimed a classification for the complex quadrics (which are symmetric spaces of rank 2) in Ref. 3, and then for all symmetric spaces of rank 2 in Ref. 4, using their construction of polars and meridians described above. However, it turns out that their classifications are incorrect: For several spaces of rank 2, totally geodesic submanifolds have been missed, and also some other details are faulty. In my papers Refs. 7–10 I discuss these shortcomings and give a full classification of the totally geodesic submanifolds in all irreducible symmetric spaces of rank 2; Section 2 of the present exposition contains a summary of these results. For symmetric spaces of rank ≥ 3, the full classification problem is still open.

However, there are several results concerning the classification of special classes of totally geodesic submanifolds. Probably the most significant result of this kind is the classification of reflective submanifolds in all Riemannian symmetric spaces due to Leung; his results are found in final form in Ref. 14, but also see Refs. 12,13. Another important problem of this kind is the classification of the totally geodesic submanifolds M' of $M = G/K$ with maximal rank (i.e. $\mathrm{rk}(M') = \mathrm{rk}(M)$); this problem has been solved for the symmetric spaces with $\mathrm{rk}(M) = \mathrm{rk}(G)$ by Ikawa/Tasaki,[6] and then for all irreducible symmetric spaces by Zhu/Liang.[20]

Further important classification results concern Hermitian symmetric spaces M: In them, the complex totally geodesic submanifolds have been classified by Ihara.[5] Moreover, the real forms of M (i.e. the totally real, totally geodesic submanifolds M' of M with $\dim_{\mathbb{R}}(M') = \dim_{\mathbb{C}}(M)$) are all reflective; due to this fact Leung was able to derive a classification of the real forms of all Hermitian symmetric spaces from his classification of reflective submanifolds.[15]

Finally we mention a result by Wolf concerning totally geodesic submanifolds M' of (real, complex or quaternionic) Grassmann manifolds $G_r(\mathbb{K}^n)$ with the property that any two distinct elements of M' have zero intersection, regarded as r-dimensional subspaces of $\mathbb{K}^n$. Wolf showed[18,19] that any such totally geodesic submanifold is isometric either to a sphere or to a projective space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$; he was also able to describe embeddings for these submanifolds explicitly and to calculate their maximal dimension in dependence of r and n.

2. Maximal totally geodesic submanifolds in the Riemannian symmetric spaces of rank 2

In the following, I list the isometry types corresponding to all the congruence classes of totally geodesic submanifolds which are maximal (i.e. not strictly contained in another connected totally geodesic submanifold) in all Riemannian symmetric spaces of rank 2. In many cases I also briefly describe totally geodesic embeddings corresponding to these submanifolds. This is a summary of my work in Refs. 7–10, where it is proved that the lists given here are complete, and where the totally geodesic embeddings are described in more detail.

The invariant Riemannian metric of an irreducible Riemannian symmetric space is unique only up to a positive constant. In the sequel, we use the following notations to describe the metric which is induced on the totally geodesic submanifolds: For $\ell \in \mathbb{N}$ and $r > 0$ we denote by $\mathbb{S}^\ell_r$ the ℓ-dimensional sphere of radius r, and for $\varkappa > 0$ we denote by $\mathbb{R}\mathrm{P}^\ell_\varkappa$, $\mathbb{C}\mathrm{P}^\ell_\varkappa$, $\mathbb{H}\mathrm{P}^\ell_\varkappa$ and $\mathbb{O}\mathrm{P}^2_\varkappa$ the respective projective spaces, their metric being scaled in such a way that the *minimal* sectional curvature is $\varkappa$. ($\mathbb{R}\mathrm{P}^\ell_\varkappa$ is then of constant sectional curvature $\varkappa$, $\mathbb{C}\mathrm{P}^\ell_\varkappa$ is of constant holomorphic sectional curvature $4\varkappa$, and we have the inclusions $\mathbb{R}\mathrm{P}^\ell_\varkappa \subset \mathbb{C}\mathrm{P}^\ell_\varkappa \subset \mathbb{H}\mathrm{P}^\ell_\varkappa$ of totally geodesic submanifolds). For symmetric spaces of rank 2, we describe the appropriate metric by stating the length a of the shortest restricted root of the space as a subscript $_{\mathrm{srr}=a}$. For the three infinite families of Grassmann manifolds $G_2^+(\mathbb{R}^n)$, $G_2(\mathbb{C}^n)$ and $G_2(\mathbb{H}^n)$, we also use the notation $_{\mathrm{srr}=1*}$ to denote the metric scaled in such a way that the shortest root occurring *for large* n has length 1, disregarding the fact that this root might vanish for certain small values of n.

The spaces in which the totally geodesic submanifolds are classified below are always taken with $_{\mathrm{srr}=1*}$ (for the Grassmann manifolds) or $_{\mathrm{srr}=1}$ (for all others).

2.1. $G_2^+(\mathbb{R}^{n+2})$

(a) $G_2^+(\mathbb{R}^{n+1})_{\mathrm{srr}=1*}$
The linear isometric embedding $\mathbb{R}^{n+1} \to \mathbb{R}^{n+2},\ (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_{n+1}, 0)$ induces a totally geodesic, isometric embedding $G_2^+(\mathbb{R}^{n+1}) \to G_2^+(\mathbb{R}^{n+2})$.

(b) $\mathbb{S}^n_{r=1}$
Fix a unit vector $v_0 \in \mathbb{R}^{n+2}$, and let $\mathbb{S} := \{\, v \in \mathbb{R}^{n+2} \,|\, \langle v, v_0 \rangle = 0,\ \|v\| = 1 \,\} \cong \mathbb{S}^n_{r=1}$. Then the map $\mathbb{S} \to G_2^+(\mathbb{R}^{n+2}),\ v \mapsto \mathbb{R}v \oplus \mathbb{R}v_0$ is

a totally geodesic, isometric embedding.

(c) $(\mathbb{S}^{\ell}_{r=1} \times \mathbb{S}^{\ell'}_{r=1})/\mathbb{Z}_2$, where $\ell + \ell' = n$
The map

$$\begin{aligned} \mathbb{S}^{\ell}_{r=1} \times \mathbb{S}^{\ell'}_{r=1} &\to G_2^+(\mathbb{R}^{n+2}) \\ ((x_0,\ldots,x_\ell),(y_0,\ldots,y_{\ell'})) &\mapsto \mathbb{R}\,(x_0,\ldots,x_\ell,0,\ldots,0) \oplus \mathbb{R}\,(0,\ldots,0,y_0,\ldots,y_{\ell'}) \end{aligned}$$

is a totally geodesic, isometric immersion, and a two-fold covering map onto its image in $G_2^+(\mathbb{R}^{n+2})$.

(d) For $n \geq 4$ even: $\mathbb{C}\mathrm{P}^{n/2}_{\varkappa=1/2}$
Let us fix a complex structure J on $\mathbb{R}^{n+2}$. Then the complex-1-dimensional linear subspaces of $(\mathbb{R}^{n+2}, J)$ are in particular real-2-dimensional oriented linear subspaces of $\mathbb{R}^{n+2}$. Therefore the complex projective space $\mathbb{P} \cong \mathbb{C}\mathrm{P}^{n/2}$ over $(\mathbb{R}^{n+2}, J)$ is contained in $G_2^+(\mathbb{R}^{n+2})$; it turns out to be a totally geodesic submanifold.

(e) For $n = 2$: $\mathbb{C}\mathrm{P}^1_{\varkappa=1/2} \times \mathbb{R}\mathrm{P}^1_{\varkappa=1/2}$
The image of the Segré embedding $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^3$ (see for example Ref. 17, p. 55f.) is a 2-dimensional complex quadric in $\mathbb{C}\mathrm{P}^3$; such a quadric is isometric to $G_2^+(\mathbb{R}^4)$. Thereby we see that $G_2^+(\mathbb{R}^4)$ is isometric to $\mathbb{C}\mathrm{P}^1_{\varkappa=1/2} \times \mathbb{C}\mathrm{P}^1_{\varkappa=1/2}$. Let C be the trace of a (closed) geodesic in $\mathbb{C}\mathrm{P}^1_{\varkappa=1/2}$; then C is isometric to $\mathbb{R}\mathrm{P}^1_{\varkappa=1/2}$, and $\mathbb{C}\mathrm{P}^1_{\varkappa=1/2} \times C$ is a totally geodesic submanifold of $\mathbb{C}\mathrm{P}^1_{\varkappa=1/2} \times \mathbb{C}\mathrm{P}^1_{\varkappa=1/2} \cong G_2^+(\mathbb{R}^4)$.

(f) For $n = 3$: $\mathbb{S}^2_{r=\sqrt{5}}$
To describe this totally geodesic submanifold, as well as similar totally geodesic submanifolds occurring in $G_2(\mathbb{C}^6)$ and $G_2(\mathbb{H}^7)$ (see Sections 2.2(g) and 2.3(f) below), we note that there is exactly one irreducible, 14-dimensional, quaternionic representation of $\mathrm{Sp}(3)$ (see Ref. 1, Chapter VI, Section (5.3), p. 269ff.). It can be constructed as follows: The vector representation of $\mathrm{Sp}(3)$ on $\mathbb{C}^6$ induces a representation of $\mathrm{Sp}(3)$ on $\bigwedge^3 \mathbb{C}^6$. This 20-dimensional representation decomposes into two irreducible components: One, 6-dimensional, is equivalent to the vector representation of $\mathrm{Sp}(3)$; the other, acting on a 14-dimensional linear space V, is the irreducible representation we are interested in.
It turns out that the restriction of the representation of $\mathrm{Sp}(3)$ on V to an $\mathrm{SO}(3)$ embedded in $\mathrm{Sp}(3)$ in the canonical way, is a real representation, and that in any real form $V_{\mathbb{R}}$ of V, two linear independent vectors are left invariant. By splitting off the subspace of V' spanned by these vectors, we get a real-5-dimensional representation $V'_{\mathbb{R}}$ of $\mathrm{SO}(3)$, which turns out to be irreducible (and equivalent to the Cartan representation $\mathrm{SO}(3) \times \mathrm{End}_+(\mathbb{R}^3)_0 \to \mathrm{End}_+(\mathbb{R}^3)_0,\ (B, X) \mapsto BXB^{-1}$). It turns out

that the corresponding action of SO(3) on $G_2^+(V'_{\mathbb{R}}) \cong G_2^+(\mathbb{R}^5)$ has exactly one totally geodesic orbit; this orbit is isometric to $\mathbb{S}^2_{r=\sqrt{5}}$.

2.2. $G_2(\mathbb{C}^{n+2})$

(a) $G_2(\mathbb{C}^{n+1})_{\mathrm{srr}=1*}$
The linear isometric embedding $\mathbb{C}^{n+1} \to \mathbb{C}^{n+2}$, $(z_1, \ldots, z_{n+1}) \mapsto (z_1, \ldots, z_{n+1}, 0)$ induces a totally geodesic, isometric embedding $G_2(\mathbb{C}^{n+1}) \to G_2(\mathbb{C}^{n+2})$.

(b) $G_2(\mathbb{R}^{n+2})_{\mathrm{srr}=1*}$
The map $G_2(\mathbb{R}^{n+2}) \to G_2(\mathbb{C}^{n+2})$, $\Lambda \mapsto \Lambda \oplus i\Lambda$ is a totally geodesic, isometric embedding.

(c) $\mathbb{C}\mathrm{P}^n_{\varkappa=1}$
Fix a unit vector $v_0 \in \mathbb{C}^{n+2}$. Then the map $\mathbb{C}\mathrm{P}((\mathbb{C}v_0)^\perp) \to G_2(\mathbb{C}^{n+2})$, $\mathbb{C}v \mapsto \mathbb{C}v \oplus \mathbb{C}v_0$ is a totally geodesic, isometric embedding.

(d) $\mathbb{C}\mathrm{P}^\ell_{\varkappa=1} \times \mathbb{C}\mathrm{P}^{\ell'}_{\varkappa=1}$ with $\ell + \ell' = n$
Let $\mathbb{C}^{n+2} = W \oplus W'$ be a splitting of $\mathbb{C}^{n+2}$ into complex-linear subspaces of dimension $\ell + 1$ resp. $\ell' + 1$. Then $\mathbb{C}\mathrm{P}(W) \times \mathbb{C}\mathrm{P}(W') \to G_2(\mathbb{C}^{n+2})$, $(\mathbb{C}v, \mathbb{C}v') \mapsto \mathbb{C}v \oplus \mathbb{C}v'$ is a totally geodesic, isometric embedding.

(e) For n even: $\mathbb{H}\mathrm{P}^{n/2}_{\varkappa=1/2}$
Let us fix a quaternionic structure τ on $\mathbb{C}^{n+2}$ (i.e. $\tau : \mathbb{C}^{n+2} \to \mathbb{C}^{n+2}$ is anti-linear with $\tau^2 = -\mathrm{id}$). Then the quaternionic-1-dimensional linear subspaces of $(\mathbb{C}^{n+2}, \tau)$ are in particular complex-2-dimensional linear subspaces of $\mathbb{C}^{n+2}$. Therefore the quaternionic projective space $\mathbb{P} \cong \mathbb{H}\mathrm{P}^{n/2}$ over $(\mathbb{C}^{n+2}, \tau)$ is contained in $G_2(\mathbb{C}^{n+2})$; it turns out that $\mathbb{P}$ is a totally geodesic submanifold of $G_2(\mathbb{C}^{n+2})$.

(f) For $n = 2$: $G_2^+(\mathbb{R}^5)_{\mathrm{srr}=\sqrt{2}}$ and $(\mathbb{S}^3_{r=1/\sqrt{2}} \times \mathbb{S}^1_{r=1/\sqrt{2}})/\mathbb{Z}_2$
Note that $G_2(\mathbb{C}^4)_{\mathrm{srr}=1*}$ is isometric to $G_2^+(\mathbb{R}^6)_{\mathrm{srr}=\sqrt{2}}$. This isometry can be exhibited via the Plücker map $G_2(\mathbb{C}^m) \to \mathbb{C}\mathrm{P}(\bigwedge^2 \mathbb{C}^m)$, $\mathbb{C}u \oplus \mathbb{C}v \mapsto \mathbb{C}(u \wedge v)$, which is an isometric embedding for any m; for $m = 4$ its image in $\mathbb{C}\mathrm{P}(\bigwedge^2 \mathbb{C}^4) \cong \mathbb{C}\mathrm{P}^5$ turns out to be a 4-dimensional complex quadric; such a quadric is isomorphic to $G_2^+(\mathbb{R}^6)$. Thus $G_2(\mathbb{C}^4)$ is isometric to $G_2^+(\mathbb{R}^6)$, hence its maximal totally geodesic submanifolds are those given in Section 2.1 for $n = 4$, namely: $G_2^+(\mathbb{R}^5)_{\mathrm{srr}=\sqrt{2}}$, $(\mathbb{S}^3_{r=1/\sqrt{2}} \times \mathbb{S}^1_{r=1/\sqrt{2}})/\mathbb{Z}_2$, $(\mathbb{S}^2_{r=1/\sqrt{2}} \times \mathbb{S}^2_{r=1/\sqrt{2}})/\mathbb{Z}_2$, $\mathbb{S}^4_{r=1/\sqrt{2}}$, $\mathbb{C}\mathrm{P}^2_{\varkappa=1}$. The first two of these submanifolds are those which are listed under this point; the remaining submanifolds have already been listed above

(note that $(\mathbb{S}^2 \times \mathbb{S}^2)/\mathbb{Z}_2$ and $\mathbb{S}^4$ are isometric to $G_2(\mathbb{R}^4)$ and $\mathbb{HP}^1$, respectively).

(g) For $n = 4$: $\mathbb{CP}^2_{\varkappa=1/5}$
Let us consider the 14-dimensional quaternionic, irreducible representation V of $\mathrm{Sp}(3)$ described in Section 2.1(f). The restriction of that representation to a $\mathrm{SU}(3)$ canonically embedded in $\mathrm{Sp}(3)$ leaves a totally complex 6-dimensional linear subspace $V_{\mathbb{C}}$ of V invariant; the resulting 6-dimensional representation $V_{\mathbb{C}}$ of $\mathrm{SU}(3)$ is irreducible. It turns out that the induced action of $\mathrm{SU}(3)$ on $G_2(V_{\mathbb{C}}) \cong G_2(\mathbb{C}^6)$ has exactly one totally geodesic orbit; this orbit is isometric to $\mathbb{CP}^2_{\varkappa=1/5}$.

2.3. $G_2(\mathbb{H}^{n+2})$

(a) $G_2(\mathbb{H}^{n+1})_{\mathrm{srr}=1*}$
The linear isometric embedding $\mathbb{H}^{n+1} \to \mathbb{H}^{n+2}$, $(q_1, \ldots, q_{n+1}) \mapsto (q_1, \ldots, q_{n+1}, 0)$ induces a totally geodesic, isometric embedding $G_2(\mathbb{H}^{n+1}) \to G_2(\mathbb{H}^{n+2})$.

(b) $G_2(\mathbb{C}^{n+2})_{\mathrm{srr}=1*}$
We fix two orthogonal imaginary unit quaternions i and j, and let $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$. Then the map $G_2(\mathbb{C}^{n+2}) \to G_2(\mathbb{H}^{n+2})$, $\Lambda \mapsto \Lambda \oplus \Lambda j$ is a totally geodesic, isometric embedding.

(c) $\mathbb{HP}^n_{\varkappa=1}$
Fix a unit vector $v_0 \in \mathbb{H}^{n+2}$. Then the map $\mathbb{HP}((v_0\mathbb{H})^\perp) \to G_2(\mathbb{H}^{n+2})$, $v\mathbb{H} \mapsto v\mathbb{H} \oplus v_0\mathbb{H}$ is a totally geodesic, isometric embedding.

(d) $\mathbb{HP}^\ell_{\varkappa=1} \times \mathbb{HP}^{\ell'}_{\varkappa=1}$ with $\ell + \ell' = n$
Let $\mathbb{H}^{n+2} = W \oplus W'$ be a splitting of $\mathbb{H}^{n+2}$ into quaternionic-linear subspaces of dimension $\ell+1$ resp. $\ell'+1$. Then $\mathbb{HP}(W) \times \mathbb{HP}(W') \to G_2(\mathbb{H}^{n+2})$, $(v\mathbb{H}, v'\mathbb{H}) \mapsto v\mathbb{H} \oplus v'\mathbb{H}$ is a totally geodesic, isometric embedding.

(e) For $n = 2$: $\mathrm{Sp}(2)_{\mathrm{srr}=\sqrt{2}}$ and $(\mathbb{S}^5_{r=1/\sqrt{2}} \times \mathbb{S}^1_{r=1/\sqrt{2}})/\mathbb{Z}_2$
Let $U \in G_2(\mathbb{H}^4)$ be given, then $U^\perp$ is the only pole corresponding to U in $G_2(\mathbb{H}^4)$. The centrosome between this pair of poles is a totally geodesic submanifold of $G_2(\mathbb{H}^4)$ which is isometric to $\mathrm{Sp}(2)$. This $\mathrm{Sp}(2)$ is also a reflective submanifold of $G_2(\mathbb{H}^4)$, the complementary reflective submanifold is isometric to $(\mathbb{S}^5_{r=1/\sqrt{2}} \times \mathbb{S}^1_{r=1/\sqrt{2}})/\mathbb{Z}_2$.

(f) For $n = 5$: $\mathbb{HP}^2_{\varkappa=1/5}$
We again consider the irreducible, quaternionic 14-dimensional representation V of $\mathrm{Sp}(3)$ introduced in Section 2.1(f); we now view V as

a quaternionic-7-dimensional linear space. The representation of $\mathrm{Sp}(3)$ on V induces an action of $\mathrm{Sp}(3)$ on the quaternionic 2-Grassmannian $G_2(V) \cong G_2(\mathbb{H}^7)$; again it turns out that this action has exactly one totally geodesic orbit, which is isometric to $\mathbb{H}\mathrm{P}^2_{\varkappa=1/5}$.

(g) For $n = 4$: $\mathbb{S}^3_{r=2\sqrt{5}}$
According to the present list, two of the maximal totally geodesic submanifolds of the 2-Grassmannian $G_2(\mathbb{H}^7)$ are isometric to $G_2(\mathbb{H}^6)$ and $\mathbb{H}\mathrm{P}^2_{\varkappa=1/5}$, respectively. The intersection of these two totally geodesic submanifolds is a totally geodesic submanifold of $G_2(\mathbb{H}^6)$, which turns out to be isometric to $\mathbb{S}^3_{r=2\sqrt{5}}$.

2.4. SU(3)/SO(3)

(a) $(\mathbb{S}^2_{r=1} \times \mathbb{S}^1_{r=\sqrt{3}})/\mathbb{Z}_2$
(b) $\mathbb{R}\mathrm{P}^2_{\varkappa=1/4}$

2.5. SU(6)/Sp(3)

(a) $\mathbb{H}\mathrm{P}^2_{\varkappa=1/4}$
(b) $\mathbb{C}\mathrm{P}^3_{\varkappa=1/4}$
(c) $\mathrm{SU}(3)_{\mathrm{srr}=1}$
The map $\mathrm{SU}(3) \to \mathrm{SU}(6)/\mathrm{Sp}(3)$, $B \mapsto \left(\begin{smallmatrix} B & 0 \\ 0 & B^{-1} \end{smallmatrix}\right) \cdot \mathrm{Sp}(3)$ is a totally geodesic embedding of this type.
(d) $(\mathbb{S}^5_{r=1} \times \mathbb{S}^1_{r=\sqrt{3}})/\mathbb{Z}_2$

2.6. SO(10)/U(5)

In the descriptions of the embeddings for this symmetric space, we consider both $\mathrm{U}(5)$ and $\mathrm{SO}(10)$ as acting on $\mathbb{C}^5 \cong \mathbb{R}^{10}$; in the latter case, this action is only $\mathbb{R}$-linear.

(a) $\mathbb{C}\mathrm{P}^4_{\varkappa=1}$
(b) $G_2(\mathbb{C}^5)_{\mathrm{srr}=1}$
(c) $\mathbb{C}\mathrm{P}^3_{\varkappa=1} \times \mathbb{C}\mathrm{P}^1_{\varkappa=1}$
Let $G := \mathrm{SO}(6) \times \mathrm{SO}(4)$ be canonically embedded in $\mathrm{SO}(10)$ in such a way that its intersection with $\mathrm{U}(5)$ is maximal. Then $G/(G \cap \mathrm{U}(5))$ is a totally geodesic submanifold of $\mathrm{SO}(10)/\mathrm{U}(5)$ which is isometric to $(\mathrm{SO}(6)/\mathrm{U}(3)) \times (\mathrm{SO}(4)/\mathrm{U}(2)) \cong \mathbb{C}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^1$.

(d) $G_2^+(\mathbb{R}^8)_{\mathrm{srr}=\sqrt{2}}$
Let $G := \mathrm{SO}(8)$ be canonically embedded in $\mathrm{SO}(10)$ in such a way that its intersection with $\mathrm{U}(5)$ is maximal. Then $G/(G \cap \mathrm{U}(5))$ is a totally geodesic submanifold of $\mathrm{SO}(10)/\mathrm{U}(5)$ which is isometric to $\mathrm{SO}(8)/\mathrm{U}(4) \cong G_2^+(\mathbb{R}^8)$.

(e) $\mathrm{SO}(5)_{\mathrm{srr}=1}$
The map $\mathrm{SO}(5) \to \mathrm{SO}(10)/\mathrm{U}(5)$, $B \mapsto \left(\begin{smallmatrix} B & 0 \\ 0 & B^{-1} \end{smallmatrix}\right) \cdot \mathrm{U}(5)$ is a totally geodesic embedding of this type.

2.7. $E_6/(\mathrm{U}(1) \cdot \mathrm{Spin}(10))$

(a) $\mathbb{O}\mathrm{P}^2_{\varkappa=1/2}$
(b) $\mathbb{C}\mathrm{P}^5_{\varkappa=1} \times \mathbb{C}\mathrm{P}^1_{\varkappa=1}$
(c) $G_2^+(\mathbb{R}^{10})_{\mathrm{srr}=\sqrt{2}}$
(d) $G_2(\mathbb{C}^6)_{\mathrm{srr}=1}$
(e) $(G_2(\mathbb{H}^4)/\mathbb{Z}_2)_{\mathrm{srr}=1}$
(f) $\mathrm{SO}(10)/\mathrm{U}(5)_{\mathrm{srr}=1}$

2.8. E_6/F_4

(a) $\mathbb{O}\mathrm{P}^2_{\varkappa=1/4}$
(b) $\mathbb{H}\mathrm{P}^3_{\varkappa=1/4}$
(c) $((\mathrm{SU}(6)/\mathrm{Sp}(3))/\mathbb{Z}_3)_{\mathrm{srr}=1}$
(d) $(\mathbb{S}^9_{r=1} \times \mathbb{S}^1_{r=\sqrt{3}})/\mathbb{Z}_4$

2.9. $G_2/\mathrm{SO}(4)$

(a) $\mathrm{SU}(3)/\mathrm{SO}(3)_{\mathrm{srr}=\sqrt{3}}$
(b) $(\mathbb{S}^2_{r=1} \times \mathbb{S}^2_{r=1/\sqrt{3}})/\mathbb{Z}_2$
(c) $\mathbb{C}\mathrm{P}^2_{\varkappa=3/4}$
(d) $\mathbb{S}^2_{r=\frac{2}{3}\sqrt{21}}$

2.10. SU(3)

(a) $\mathrm{SU}(3)/\mathrm{SO}(3)_{\mathrm{srr}=1}$
The Cartan embedding $f : \mathrm{SU}(3)/\mathrm{SO}(3) \to \mathrm{SU}(3)$ is a totally geodesic embedding of this type.

(b) $(\mathbb{S}^3_{r=1} \times \mathbb{S}^1_{r=\sqrt{3}})/\mathbb{Z}_2$

(c) $\mathbb{C}\mathrm{P}^2_{\varkappa=1/4}$
The Cartan embedding $f : \mathrm{SU}(3)/\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1)) \to \mathrm{SU}(3)$ is a totally geodesic embedding of this type.

(d) $\mathbb{R}\mathrm{P}^3_{\varkappa=1/4}$

2.11. Sp(2)

(a) $G_2^+(\mathbb{R}^5)_{\mathrm{srr}=1}$
The Cartan embedding $f : \mathrm{Spin}(5)/(\mathrm{Spin}(2)\times\mathrm{Spin}(3)) \to \mathrm{Spin}(5) \cong \mathrm{Sp}(2)$ is a totally geodesic embedding of this type.

(b) $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$
The canonically embedded $\mathrm{Sp}(1)\times\mathrm{Sp}(1) \subset \mathrm{Sp}(2)$ is a totally geodesic submanifold of this type.

(c) $\mathbb{H}\mathrm{P}^1_{\varkappa=1/2}$
The Cartan embedding $f : \mathrm{Sp}(2)/(\mathrm{Sp}(1)\times\mathrm{Sp}(1)) \to \mathrm{Sp}(2)$ is a totally geodesic embedding of this type.

(d) $\mathbb{S}^3_{r=\sqrt{5}}$

2.12. G_2

(a) $G_2/\mathrm{SO}(4)_{\mathrm{srr}=1}$
The Cartan embedding $f : G_2/\mathrm{SO}(4) \to G_2$ is a totally geodesic embedding of this type.

(b) $(\mathbb{S}^3_{r=1}\times\mathbb{S}^3_{r=1/\sqrt{3}})/\mathbb{Z}_2$

(c) $\mathrm{SU}(3)_{\mathrm{srr}=\sqrt{3}}$
Regard G_2 as the automorphism group of the division algebra of the octonions $\mathbb{O}$ and fix an imaginary unit octonion i. Then the subgroup $\{\, g \in G_2 \,|\, g(i) = i \,\}$ is isomorphic to $\mathrm{SU}(3)$ and a totally geodesic submanifold of this type.

(d) $\mathbb{S}^3_{r=\frac{2}{3}\sqrt{21}}$

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